Complex chaos in the conditional dynamics of qubits

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We analyse the consequences of iterative measurement-induced non-linearity on the dynamical behaviour of qubits. We present a one-qubit scheme where the equation governing the time evolution is a complex-valued non-linear map with one complex parameter. In contrast to the usual notion of quantum chaos, exponential sensitivity to the initial state occurs here. We calculate analytically the Lyapunov exponent based on the overlap of quantum states, and find that it is positive. We present a few illustrative examples of the emerging dynamics.

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Exponential sensitivity to inital conditions in nonlinear systems, first described by Poincaré [1], is today known as chaos. Quantum systems which are classically chaotic generally do not show exponential sensitivity even if their dynamics are complicated and refered to as quantum chaos [2]. Unitarity of the evolution generally prohibits exponential sensitivity, although rather exotic exceptions were found [3].

Measuring a quantum system affects its dynamics. As a result, instead of the original unitary evolution an effective nonlinear dynamics may emerge. This phenomeon was investigated extensively in quantum systems with continuous degrees of freedom which possess a corresponding classical limit [4]. In particular, it was demonstrated that this sensitivity is strong in dynamical regimes in which relevant actions are large in comparison with Planck's constant and the corresponding classical dynamics are chaotic. The continuously monitored system evolves in such cases according to a stochastic nonlinear Schrödinger equation, reflecting the fact that measurements have non-deterministic output. Very recently, Habib, Jacobs and Shizume could prove [5] that a continuously measured classically chaotic system can be truly chaotic even far from any classical limit. They numerically calculated the Lyapunov exponent and found that it is positive for the expectation value of the position operator.

An alternative way of handling measurement results is to use them as conditions and select the subensemble according to the prescribed output, thus evoking a deterministic effective dynamics for the subensemble. Conditional dynamics is of considerable current interest in the context of quantum information science for qubit systems. In quantum state purification protocols [6], for example, such non-linear effects are exploited to guarantee the unconditional security of quantum cryptographic key distribution protocols [7]. Though basic aspects of measurement-induced non-linear effects have already been investigated in these systems, it is still largely unexplored whether these effects may lead to chaos and, in case they do, what their characteristic dynamical features are.

In this paper we investigate the dynamical features of iterated deterministic quantum maps which describe the measurement-induced conditional dynamics of one- and two qubit systems and which have been proposed recently in the context of quantum information [8]. These qubit systems do not have a (trivial) classical limit. The idea that feeding results from weak measurements back into the dynamics of an ensemble of quantum systems could possibly lead to a novel type of quantum chaos with sensitivity to initial conditions was mentioned already by Lloyd and Slotine [11]. The present scheme is related to this idea although it is based not on weak but on standard strong selective measurements. The selection, conditioned on measurement outcomes on part of the system, can be thought of as feeding information back into the remaining subensemble.

As a main result it will be demonstrated that even in the simplest possible case of one-qubit systems the resulting dynamics of pure quantum states are governed by a special class of non-linear maps in one complex-valued variable. First studies of the iterative behavior of such complex-valued non-linear maps were performed already a century ago by Fatou [9] thus paving the way for a new research field within non-linear dynamics [10]. Therefore, a detailed understanding of the sensitivity of these measurement-induced quantum maps with respect to initial states can be obtained by taking advantage of the concepts and theorems from the so-far unrelated theory of iterated maps with one complex variable. In order to demonstrate the richness inherent in this 'complex chaos' a few illustrative examples are presented.

Consider the nonlinear quantum transformation [8]:

$$\rho' = \mathcal{S}\rho\,, \quad \rho_{ij} \xrightarrow{\mathcal{S}} N\rho_{ij}^2 \tag{1}$$

with the renormalization factor $N = 1/\sum \rho_{ii}^2$. Thereby, the matrix element squaring is defined with respect to a prescribed orthonormal basis $\{|i\rangle\}$. In the special case of qubits this deterministic non-linear map can be realized by applying a controlled-not gate on a pair of equally prepared qubits and then filtering by a measurement performed on one of them [8]. By repeating the transforma-

tion $\mathcal{S}$ we expect that smaller diagonal matrix elements will tend to zero and the largest one survives, converging to unity. In other words, pure states can be stable fixed points of the map, leading to purification of the state. The maximally mixed state is also a fixed point of the map $\mathcal{S}$, but it is easy to see that it is an unstable one. Perturbing the initial state by increasing the weight of one of the states in the mixture will lead to purification towards that particular state.

While the squaring operator itself behaves rather simply, the dynamics become highly nontrivial if an additional local unitrary transformation, a rotation in the qubit Hilbert space, is applied

$$\mathcal{R}\rho = U\rho U^\dagger\,, \tag{2}$$

with

$$U = \begin{pmatrix} \cos x & \sin x\; e^{i\phi} \\ -\;\sin x\; e^{-i\phi} & \cos x \end{pmatrix}, \tag{3}$$

in the prescribed basis. In this way one step of the dynamics reads

$$\rho' = \mathcal{F}\rho = \mathcal{R}\mathcal{S}\rho\,, \tag{4}$$

and repeating the transformation $\mathcal{F}$ leads to the discrete conditional time-evolution we are interested in. It should be kept in mind that each step of this particular deterministic time evolution consists of a quantum mechanical filtering process involving two qubits. As a result of this filtering process either the target qubit or both qubits have to be dismissed. The size of the original ensemble decreases exponentially in time.

Let us first consider an initial pure state of the qubit. With the notation

$$|\psi\rangle = N(z|0\rangle + |1\rangle)\,, \tag{5}$$

where the state is normalized by $N = (1+|z|^2)^{-1/2}$, the single complex parameter z describes the state of the qubit. The transformation $\mathcal{F}$ maps this pure state onto a pure state and transforms z as

$$z \mapsto F_p(z) = \frac{z^2+p}{1-p^* z^2}\,, \tag{6}$$

where $p = \tan x e^{i\phi}$ and the star denotes complex conjugation. The conditional dynamics of the qubit are thus governed by $F_p(z)$, which is a non-linear $\mathbb{C} \to \mathbb{C}$ map with one complex parameter p. A considerable difference compared to chaotic systems in classical physics is that the underlying space is complex, here. Even the simplest non-linear maps of the complex plane can show intricate dynamical structure, such as the famous Mandelbrot set. The study of the mathematics related to maps in one complex variable has a long history and an extensive literature, (for a review see [10]). The map (6) is a rational function of second order polynomials, similar to the one first studied by Fatou [9] a century ago: $z \mapsto z^2/(z^2+2)$.

The traditional approach to a non-linear map in one complex variable is to divide the complex plane of the initial values z_0 into regular and irregular points forming the Fatou and Julia sets, respectively. Regular starting points from the Fatou set will converge to a stable cycle (also elements of the Fatou set) when repeating the iteration. Taking into account both the initial condition z_0 and the complex parameter p a four dimensional parameter space is defined. We will select special parameter values when studying the map in order to illustrate the richness of its behaviour and demonstrate sensitivity to the initial conditions.

The full dynamics induced by Eq. (6) take place on the Riemann sphere $\hat{\mathbb{C}}$ consisting of $\mathbb{C}$ together with the point at infinity. The physical meaning of the points 0 and ∞ for z are the two basis states of the qubit, $|1\rangle$ and $|0\rangle$, respectively. The map $F_p(z)$ is a rational function of degree two. A general theorem on rational maps that are quotients of two polynomials ensures that the Julia set is not empty. The non-vacuous Julia set is the usual condition for complex valued maps to be considered chaotic. A more subtle question is whether the map is hyperbolic, *i.e.* expanding on the Julia set. The latter property is closer to the sense how the term chaos is used for dynamical systems. For rational maps of degree two this can be decided by following the orbit of the critical points: each orbit should converge to some attractive periodic orbit. For definition and a review of the above properties we suggest to consult Milnor's book [10].

The problem simplifies considerably if the parameter p is set to zero. The map is then the simple squaring $F_0(z) = z^2$. This is a well-known example of a simple Julia set formed by the unit circle in the complex plane. If the starting point is within the unit disk, the iterations converge to zero, while initial values from outside converge to infinity. Initial points with absolute value exactly one will not converge, but the iterations follow an irregular path on the unit circle. In other words, the qubit will tend to one of the basis states, except for the equally weighted linear superposition. In the latter case the relative phase will follow irregular dynamics. The Julia set (a circle) is one-dimensional. We use the following definition of the Lyapunov exponent λ

$$\lambda = \lim_{n\to\infty}\ \lim_{\Delta(0)\to 0}\ \frac{1}{n}\ln\frac{\Delta(n)}{\Delta(0)}\,, \tag{7}$$

where we choose Δ to be the distance related to the overlap of the two corresponding quantum states: $\Delta = 1 - |\langle\psi_1|\psi_2\rangle|^2$. For unitary evolution this is a constant quantity, hence the above defined Lyapunov exponent of a closed quantum system is always zero. It is generally not straightforward to apply this definition, since the result of the limit may depend on the path in the Hilbert space one takes when approaching the two initial states towards each other. Nevertheless, in our simple case we can restrict ourself to the unit circle. On this one dimensional manifold we may define the Lyapunov exponent with respect to the phase variable by choosing both ini-

tial states with $|z| = 1$. Without loss of generality we can take $z_0 = 1$, $z_1 = e^{i\varphi}$ and arrive at

$$\lambda_\varphi = \lim_{n\to\infty} \lim_{\varphi\to 0} \frac{1}{n} \ln \frac{\Delta_\varphi(n)}{\Delta_\varphi(0)} \,, \tag{8}$$

with $\Delta_\varphi(n) = \frac{1}{2}(1 - \cos 2^n \varphi)$. The order of limits is important and therefore, we can *first* let φ go to zero, thus we can use the Taylor expansion of the cosine in φ which leads to the result

$$\lambda_\varphi = 2 \ln 2 \,. \tag{9}$$

The positive Lyapunov exponent indicates that exponentially fast separation takes place in the Hilbert space of the system.

Non-zero fixed values of p are expected to lead to qualitatively different dynamics. The corresponding Julia sets can possess highly non-trivial structures. By solving the nth order eigenvalue equation $F_p^{\circ n}(z) = z$ (where $F^{\circ n}(z)$ denotes n times repeated action of F), one can find various-order periodic cycles $\{z_1, z_2, \ldots, z_n\}$, $(z_i \neq z_j)$. The stability of a cycle can then be determined by evaluating the multiplier $\lambda = F_p'(z_1)F_p'(z_2)\ldots F_p'(z_n)$. If the absolute value of this multiplier is smaller then unity, the fixed point is attractive. If it is greater than unity it is repelling, while for $|\lambda| = 1$ it is neutral. Those neutral periodic cycles for which λ is a root of unity, and for which no iterate of the map is the identity are called parabolic. A rational map of degree two can have at most two cycles which are attracting or neutral [10]. While the first order fixed points are given by a third order equation, in general the period-n fixed points would require the solution of a polynomial equation of order $2^n + 1$. The critical points z_c of a map, where $F'(z_c) = 0$, play a special role in the theory of non-linear maps. For the map F_p the two critical points are $z_{c1} = 0$ and $z_{c2} = \infty$, independent of p. The orbits of the two critical points characterize a rational map to a large extent [12]. By checking the convergence of their orbit to an attracting periodic cycle, one can decide whether the map is hyperbolic. Moreover, all attractive and parabolic cycles can be found in this way.

Physically speaking, the parameter p describes the rotation of the qubit state $|\psi\rangle$. Setting the parameter to $p = 1$ corresponds to a rotation of $\pi/4$ that transforms, for example, the basis states into their equal superpositions. This is a symmetric situation with respect to the basis states. The orbit of one critical point, $z_{c2} = \infty$, is part of the attractive cycle $\{-1, \infty\}$. The other critical point $z_{c1} = 0$ follows the orbit $0 \mapsto 1 \mapsto \infty$ and thus lands on the same periodic cycle. Therefore the only stable cycle for this map is the fixed point $\{-1, \infty\}$. This also proves that the map is hyperbolic [10]. Numerical calculation of the Julia set for quadratic rational maps is difficult. There are no general algorithms to compute it. Here we can simply apply the criterion of convergence to the stable cycle when plotting the Julia set in Fig.(1) for $p = 1$. Dark points converge fast, gray points slower,

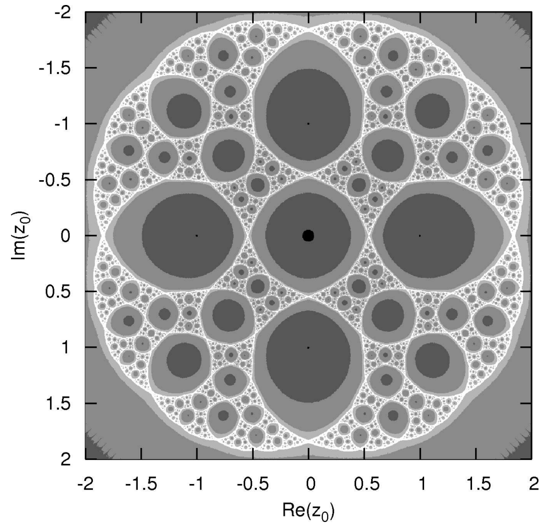

FIG. 1: The Julia set for the non-linear map (6). The parameter is set to $p = 1$. Grayscale indicates how fast the map converges to the stable cycle $\{-1, \infty\}$ (dark – fast, gray – slow convergence, white – no convergence).

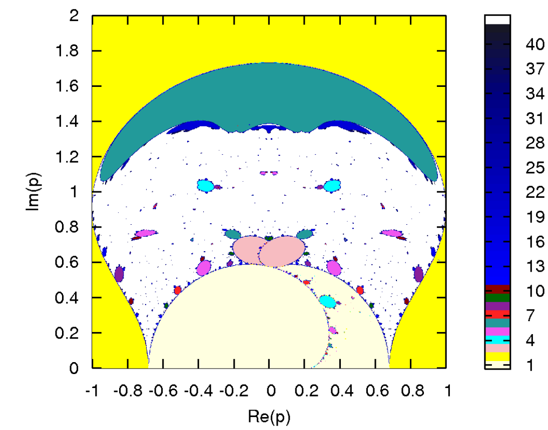

FIG. 2: (Color online) The complex parameter space p of the non-linear map (6), with the initial state being the critical point $z_0 = 0$. Colors indicate the length of attractive cycles. White corresponds to no convergence.

white points do not converge. The complicated (fractal) structure of the Julia set reflects the sensitivity of the dynamics to the initial state: a change in the initial state may alter the dynamics from regular to chaotic and this can occur on arbitrarily small scales.

The family of maps with varying values of $p \in \mathbb{C}$ may possess fixed cycles of various length, but only at most two of them can be attractive. The critical points converge to these attractive cycles, if their orbit is convergent at all [10]. Fig. (2) depicts the complex plane of p-values with colors showing the length of the stable periodic cycle starting from the critical point $z_0 = 0$. The lower half plane is not shown, since it is a mirror image of the upper one. The attractive periodic cycles are visualized for a fixed real value of p in Fig. (3) by showing the absolute

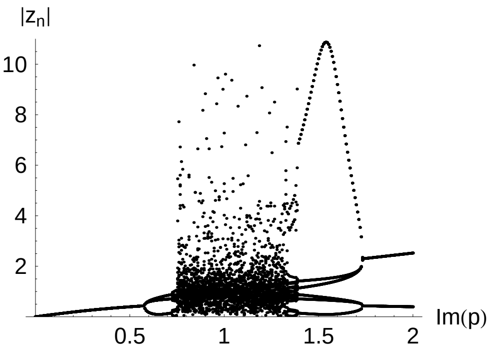

FIG. 3: Iterations of the non-linear map (6) for purely imaginary p with the initial state being the critical point $z_0 = 0$. After 10^4 iterations the absolute values of z for the next 50 steps are shown.

value of z after several iterations starting from $z_0 = 0$.

While the single qubit case well serves the purpose of rigorously demonstrating the presence of complex chaos, two qubit systems with conditional dynamics are of considerable practical interest for quantum state purification. As the mathematical form of the procedure of purification is similar in essence to the single-qubit case we expect also similar dynamical properties for two-qubit systems. In particular, parameter ranges and initial states should exist for which purification protocols exhibit true chaos. In order to address this question let us consider the following iteration acting on two-qubit states [8]

$$\rho' = \mathcal{F}\rho = \mathcal{R}_1\mathcal{R}_2\mathcal{S}\rho\,. \tag{10}$$

Thereby, $\mathcal{S}$ is the element squaring defined in Eq. (1) with the index i running from 1 to 4 through the elements of the product basis of the qubits $\{|j\rangle|k\rangle\}$, $(j, k = 0, 1)$ and the rotation $\mathcal{R}_m$ acts on the mth qubit, with parameters $x_m\,, \phi_m$ as defined in Eqs. (2,3). Now, the parameters of the two (local) complex rotations span $\mathbb{C}^2$, and the initial state can be any valid two-qubit density operator. Obviously, this is an even much larger parameter space to explore, which includes the one-qubit pure states as a special case. Our numerical simulations indicate that meta-stable purification can occur here. An initial state with some deviation from a target pure state is being purified though several iterations, but then suddenly stability is lost and chaos sets in [13].

As an application, one could try to exploit the sensitivity of the system and use it as a Schrödinger microscope [11]. Tuning into a regime where a few iterations amplify initial small differences of states one could distinguish states exponentially fast. The cost paid is the exponential size of the equally prepared systems needed. To understand the general conditions which allow exponential sensitivity to initial states would be of use for any protocol applying measurement conditioned selection.

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